

An extension of the Bernoulli polynomials inspired by the Tsallis statistics

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Abstract

In [1, 2] Carlitz introduced the degenerate Bernoulli numbers and polynomials by replacing the exponential factors in the corresponding classical generating functions with their deformed analogs: $\exp(t) \rightarrow (1 + \lambda t)^{1/\lambda}$, and $\exp(tx) \rightarrow (1 + \lambda t)^{x/\lambda}$. The deformed exponentials reduce to their ordinary counterparts in the $\lambda \rightarrow 0$ limit. In the present work we study the extension of the Bernoulli polynomials obtained via an alternate deformation $\exp(tx) \rightarrow (1 + \lambda tx)^{1/\lambda}$ that is inspired by the concepts of q -exponential function and q -logarithm used in the nonextensive Tsallis statistics.

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I Introduction

The well-known generating functions for the standard Bernoulli numbers $\{B_n\}_{n \geq 0}$ and Bernoulli polynomials $\{B_n(x)\}_{n \geq 0}$ are, respectively, given by

$$\frac{t}{\exp(t) - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad \frac{t \exp(tx)}{\exp(t) - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad B_n(0) = B_n. \quad (1.1)$$

Modifying the above generating functions via the replacement $\exp(t) \rightarrow (1 + \lambda t)^{1/\lambda}$ for $\lambda \neq 0$, Carlitz introduced [1, 2] the degenerate Bernoulli numbers $\{\beta_n(\lambda)\}_{n \geq 0}$

$$\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda) \frac{t^n}{n!}, \quad (1.2)$$

and the corresponding Bernoulli polynomials $\{\beta_n(\lambda, x)\}_{n \geq 0}$ as follows:

$$\frac{t(1 + \lambda t)^{x/\lambda}}{(1 + \lambda t)^{1/\lambda} - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}, \quad \beta_n(\lambda, 0) = \beta_n(\lambda). \quad (1.3)$$

Obviously, in the $\lambda \rightarrow 0$ limit the degenerate Bernoulli numbers and polynomials (1.2, 1.3) reduce, respectively, to their classical analogs.

On the other hand, the nonextensive statistical mechanics, pioneered by Tsallis [3], is based on a deformed exponential function, called the Tsallis q -exponential function, defined as

$$\exp_q(\mathcal{X}) = (1 + (1 - q)\mathcal{X})^{1/(1-q)}, \quad \lim_{q \rightarrow 1} \exp_q(\mathcal{X}) = \exp(\mathcal{X}). \quad (1.4)$$

The deformed exponential (1.4) may be inverted via the corresponding q -logarithm:

$$\log_q(\mathcal{X}) = \frac{\mathcal{X}^{1-q} - 1}{1 - q}, \quad \log_q(\exp_q(\mathcal{X})) = \mathcal{X}. \quad (1.5)$$

Identifying the parameters $(1 - q) \rightarrow \lambda$ it readily follows that the Tsallis q -exponential function (1.4) is precisely same as the modification introduced by Carlitz to construct the generating relations (1.2, 1.3). For our purpose, we slightly alter the notations given in (1.4, 1.5):

$$\exp_\lambda(\mathcal{X}) = (1 + \lambda \mathcal{X})^{1/\lambda}, \quad \log_\lambda(\mathcal{X}) = \frac{1}{\lambda}(\mathcal{X}^\lambda - 1), \quad (1.6)$$

where the inversion relation reads: $\log_\lambda(\exp_\lambda(\mathcal{X})) = \mathcal{X}$. The inverse relationship between the λ -logarithm and the λ -exponential functions are, however, not utilized in the choice (1.3):

$$\log_\lambda((\exp_\lambda(t))^x) = \frac{1}{\lambda}((1 + \lambda t)^x - 1) = \sum_{n=1}^{\infty} \lambda^{n-1} (x)_n \frac{t^n}{n!}, \quad (1.7)$$

where the falling factorial reads $(x)_n = x(x-1)(x-2) \cdots (x-n+1)$. On the other hand, the alternate deformation

$$\exp(tx) \rightarrow \exp_\lambda(tx) \quad (1.8)$$

employs the inverse functional property: $\log_\lambda(\exp_\lambda(tx)) = tx$. The above discussion illustrates that if we replace $\exp(tx)$ in the generating function (1.1) by its λ -analog then the two substitutes, namely, $(\exp_\lambda(t))^x$ and $\exp_\lambda(tx)$, despite both of them agreeing in the limit $\lambda \rightarrow 0$, lead to drastically different properties. Contrasting the choice followed in [2], we, in this work, explore some of the consequences of introducing the other deformation (1.8) in the generating relation (1.1) for the Bernoulli polynomials.

II $\tilde{\beta}$ -Bernoulli polynomials

A. Generating function

We introduce the deformed Bernoulli polynomials $\{\tilde{\beta}_n(\lambda|x)\}_{n \geq 0}$ by altering the generating function (1.3) as follows:

$$\frac{t \exp_\lambda(tx)}{\exp_\lambda(t) - 1} = \sum_{n=0}^{\infty} \tilde{\beta}_n(\lambda|x) \frac{t^n}{n!}, \quad \tilde{\beta}_n(\lambda|0) = \beta_n(\lambda), \quad \lim_{\lambda \rightarrow 0} \tilde{\beta}_n(\lambda|x) = B_n(x). \quad (2.1)$$

Substituting the binomial expansion

$$(\exp_\lambda(\mathcal{X}))^{\pm 1} = \sum_{n=0}^{\infty} (\pm 1)^n \varepsilon_\lambda^{(\mp)}(n) \frac{\mathcal{X}^n}{n!}, \quad \varepsilon_\lambda^{(\pm)}(n) = \begin{cases} 1, & \text{for } n = 0, \\ \prod_{j=0}^{n-1} (1 \pm j\lambda), & \text{for } n \geq 1 \end{cases} \quad (2.2)$$

in the generating function (2.1), we obtain

$$\sum_{n=0}^{\infty} \varepsilon_\lambda^{(-)}(n) \frac{t^n x^n}{n!} = \left(\sum_{k=0}^{\infty} \varepsilon_\lambda^{(-)}(k+1) \frac{t^k}{(k+1)!} \right) \left(\sum_{\ell=0}^{\infty} \tilde{\beta}_\ell(\lambda|x) \frac{t^\ell}{\ell!} \right). \quad (2.3)$$

Equating terms $O(t^n)$ on both sides in (2.3) we now produce the expansion of x^n via the $\tilde{\beta}_n(\lambda|x)$ polynomials:

$$\varepsilon_\lambda^{(-)}(n) x^n = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} \varepsilon_\lambda^{(-)}(n+1-k) \tilde{\beta}_k(\lambda|x), \quad n \geq 0. \quad (2.4)$$

The above equation directly furnishes the recurrence formula:

$$\tilde{\beta}_n(\lambda|x) = \varepsilon_\lambda^{(-)}(n) x^n - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} \varepsilon_\lambda^{(-)}(n+1-k) \tilde{\beta}_k(\lambda|x). \quad (2.5)$$

The generating function (2.1) also provides an explicit construction of the $\tilde{\beta}_n(\lambda|x)$ polynomials in terms of the λ -Bernoulli numbers given in (1.2):

$$\tilde{\beta}_n(\lambda|x) = \sum_{k=0}^n \binom{n}{k} \varepsilon_\lambda^{(-)}(k) \beta_{n-k}(\lambda) x^k, \quad n \geq 0. \quad (2.6)$$

It is interesting to note that, *à la* Carlitz [4], the above expansion may be symbolically expressed as

$$\tilde{\beta}_n(\lambda|x) = (\beta(\lambda) + \varepsilon_\lambda^{(-)} x)^n, \quad (2.7)$$

where it is understood that after expansion of the right member we replace $\beta(\lambda)^k$ with $\tilde{\beta}_k(\lambda)$, and $\varepsilon_\lambda^{(-)k}$ with $\varepsilon_\lambda^{(-)}(k)$. Following the above expansion we now list the first few $\tilde{\beta}_n(\lambda|x)$ polynomials as follows:

$$\begin{aligned}\tilde{\beta}_0(\lambda|x) &= 1, \quad \tilde{\beta}_1(\lambda|x) = x + \frac{1}{2}(\lambda - 1), \quad \tilde{\beta}_2(\lambda|x) = \varepsilon_\lambda^{(-)}(2)x^2 + (\lambda - 1)x - \frac{1}{6}(\lambda^2 - 1), \\ \tilde{\beta}_3(\lambda|x) &= \varepsilon_\lambda^{(-)}(3)x^3 - \frac{3}{2}(1 - \lambda)^2x^2 + \frac{1}{2}(1 - \lambda^2)x - \frac{\lambda}{4}(1 - \lambda^2), \\ \tilde{\beta}_4(\lambda|x) &= \varepsilon_\lambda^{(-)}(4)x^4 - 2(1 - \lambda)^2(1 - 2\lambda)x^3 + (1 - \lambda)(1 - \lambda^2)x^2 - \lambda(1 - \lambda^2)x \\ &\quad - \frac{1}{30}(1 - 20\lambda^2 + 19\lambda^4).\end{aligned}\tag{2.8}$$

Proceeding further we now develop a translation of the variable of the $\tilde{\beta}_n(\lambda|x)$ polynomials. Towards this we note that the generating function (2.1) may be written in a factorized form:

$$\frac{t \exp_\lambda(t(x + y))}{\exp_\lambda(t) - 1} = \frac{t \exp_\lambda(tx)}{\exp_\lambda(t) - 1} \frac{\exp_\lambda(t(x + y))}{\exp_\lambda(tx)}.\tag{2.9}$$

Utilizing the binomial expansion (2.2) we obtain the desired expression for a translational shift in the variable:

$$\tilde{\beta}_n(\lambda|x + y) = \sum_{k, \ell=0}^n (-1)^k \binom{n}{k, \ell, n - k - \ell} \varepsilon_\lambda^{(+)}(k) \varepsilon_\lambda^{(-)}(\ell) \tilde{\beta}_{n-k-\ell}(\lambda|x) x^k (x + y)^\ell, \tag{2.10}$$

where the multinomial coefficient is given by $\binom{n}{k, \ell, n-k-\ell} = \frac{n!}{k! \ell! (n-k-\ell)!}$. We observe that a contraction in the variables $x \rightarrow 0, y \rightarrow x$ in (2.10) readily yields the expansion (2.6) of the $\tilde{\beta}_n(\lambda|x)$ polynomials directly obtained earlier. Moreover, the classical limit $\lambda \rightarrow 0$ of the translation property (2.10) may be realized by noting the reduction of the coefficients $\varepsilon_\lambda^{(\pm)}(n)|_{\lambda \rightarrow 0} \rightarrow 1$. Implementation of the above limits in (2.10) now leads to the well-known shift property of the Bernoulli polynomials:

$$B_n(x + y) = \sum_{\ell=0}^n \binom{n}{\ell} B_{n-\ell}(x) y^\ell. \tag{2.11}$$

B. λ -Appell sequence

The sequence $\{P_n(x)\}_{n \geq 0}$ of Appell polynomials [5] satisfy the property

$$f(t) \exp(xg(t)) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{\phi(0) \dots \phi(n)}, \quad f(0) \neq 0, g(0) = 0, g'(0) \neq 0, \tag{2.12}$$

where $\phi : \mathbb{N} \rightarrow \mathbb{C}/\{0\}$ is an arbitrary function. The generating function introduced in (2.1) suggests a slight generalization of the above relation. In particular, we replace the exponential function in the l.h.s. of (2.1) possessing a factorized form of the argument with a more general construct:

$$\exp(xg(t)) \rightarrow \exp(\mathfrak{g}(x, t)), \text{ where } \mathfrak{g}(x, 0) = 0, \frac{d\mathfrak{g}(x, t)}{dt}|_{t=0} \neq 0. \tag{2.13}$$

As the λ -deformed exponential admits an infinite product representation

$$\exp_\lambda(tx) = \exp \left(\sum_{\ell=0}^{\infty} (-1)^\ell \lambda^\ell \frac{(tx)^{\ell+1}}{\ell+1} \right), \quad (2.14)$$

the generating function structure (2.1) provides an example of the generalized Appell sequence discussed above.

The Bernoulli polynomials form a well-known Appell sequence. As a consequence it obeys the recursive property: $B'_n(x) = nB_{n-1}(x)$ that we attempt to generalize now. Towards this end we first note the eigenfunction structure

$$\mathcal{D}_\lambda(x) \exp_\lambda(tx) = t \exp_\lambda(tx), \quad \mathcal{D}_\lambda(x) \equiv (1 + \lambda tx) \frac{d}{dx} \quad (2.15)$$

and subsequently operate both sides of the defining relation for the $\tilde{\beta}$ -Bernoulli polynomials (2.1) from the left by the composite derivative $\mathcal{D}_\lambda(x)$:

$$(1 + \lambda tx) \sum_{n=0}^{\infty} \tilde{\beta}'_n(\lambda|x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \tilde{\beta}_n(\lambda|x) \frac{t^{n+1}}{n!}. \quad (2.16)$$

Employing $\tilde{\beta}'_0(\lambda|x) = 0$, the above equation may be recast as

$$\sum_{n=1}^{\infty} \tilde{\beta}'_n(\lambda|x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} n \tilde{\beta}_{n-1}(\lambda|x) \frac{t^n}{n!} - \lambda x \sum_{n=1}^{\infty} n \tilde{\beta}'_{n-1}(\lambda|x) \frac{t^n}{n!}. \quad (2.17)$$

This leads to the desired result:

$$\tilde{\beta}'_n(\lambda|x) = n(\tilde{\beta}_{n-1}(\lambda|x) - \lambda x \tilde{\beta}'_{n-1}(\lambda|x)) \quad (2.18)$$

that may be regarded as the defining characteristic of the λ -Appell sequence. In the classical $\lambda \rightarrow 0$ limit it reduces to the standard structure. The recurrence relation (2.18) admits a series solution for the derivatives of the polynomials $\tilde{\beta}_n(\lambda|x)$:

$$\tilde{\beta}'_0(\lambda|x) = 0, \quad \tilde{\beta}'_{n+1}(\lambda|x) = (n+1)! \sum_{k=0}^n \frac{(-\lambda x)^{n-k}}{k!} \tilde{\beta}_k(\lambda|x). \quad (2.19)$$

Equation (2.18) shows that the $\tilde{\beta}$ -Bernoulli polynomials form a λ -Appell sequence constructed in the following way:

$$\tilde{\beta}_n(\lambda|x) = \tilde{\beta}_n(\lambda|0) + n \int_0^x (\tilde{\beta}_{n-1}(\lambda|u) - \lambda u \tilde{\beta}'_{n-1}(\lambda|u)) du, \quad n = 1, 2, \dots \quad (2.20)$$

In general, a sequence of polynomials $\{P_n(\lambda|x)_{n \geq 0}\}$ may be said to form a λ -Appell sequence if

$$P_n(\lambda|x) = P_n(\lambda|0) + n \int_0^x (P_{n-1}(\lambda|s) - \lambda s P'_{n-1}(\lambda|s)) ds, \quad n = 1, 2, \dots \quad (2.21)$$

For example, letting $P_n(\lambda|0) = 0$, we see that $\{\varepsilon_\lambda(n)x^n, n = 0, 1, 2, \dots\}$ form a λ -Appell sequence.

C. Determinant structure

Recently a determinantal approach for the Bernoulli polynomials $B_n(x)$ has been proposed [6, 7]. The construction employs an upper Hessenberg matrix [8] with the entries $h_{j,\ell} = 0$ if $j - \ell \geq 2$. The determinant of an upper Hessenberg matrix of order n

$$H_n = \begin{vmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & \cdots & \cdots & h_{1,n} \\ h_{2,1} & h_{2,2} & h_{2,3} & \ddots & \ddots & \ddots & h_{2,n} \\ 0 & h_{3,2} & h_{3,3} & \ddots & \ddots & \ddots & h_{3,n} \\ \vdots & 0 & h_{4,3} & h_{4,4} & \ddots & \ddots & h_{4,n} \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & h_{n,n-1} & h_{n,n} \end{vmatrix} \quad (2.22)$$

obeys the following recursive relation [7]:

$$H_0 \equiv 1, \quad H_{n(\geq 1)} = \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} \mathcal{Q}_\ell^{n-2} h_{\ell+1,n} H_\ell, \quad (2.23)$$

where the factorized coefficients read: $\mathcal{Q}_\ell^k = \prod_{j=\ell}^k h_{j+2,j+1}$, $\mathcal{Q}_{\ell(>k)}^k \equiv 1$. Employing the above recurrence relation we now provide the representation of the polynomials $\tilde{\beta}_n(\lambda|x)$ via a $(n+1) \times (n+1)$ determinant. Towards this end we define

$$\tilde{\beta}_0(\lambda|x) = 1, \quad \tilde{\beta}_{n(\geq 1)}(\lambda|x) = \frac{(-1)^n}{(n-1)!} \mathbf{D}_n(\lambda|x), \quad (2.24)$$

where the determinantal function $\mathbf{D}_n(\lambda|x)$ reads

$$\begin{vmatrix} 1 & \varepsilon_\lambda^{(-)}(1)x & \varepsilon_\lambda^{(-)}(2)x^2 & \varepsilon_\lambda^{(-)}(3)x^3 & \cdots & \varepsilon_\lambda^{(-)}(n-1)x^{n-1} & \varepsilon_\lambda^{(-)}(n)x^n \\ \varepsilon_\lambda^{(-)}(1) & \varepsilon_\lambda^{(-)}(2)\frac{1}{2} & \varepsilon_\lambda^{(-)}(3)\frac{1}{3} & \varepsilon_\lambda^{(-)}(4)\frac{1}{4} & \cdots & \varepsilon_\lambda^{(-)}(n)\frac{1}{n} & \varepsilon_\lambda^{(-)}(n+1)\frac{1}{n+1} \\ 0 & \varepsilon_\lambda^{(-)}(1) & \varepsilon_\lambda^{(-)}(2) & \varepsilon_\lambda^{(-)}(3) & \cdots & \varepsilon_\lambda^{(-)}(n-1) & \varepsilon_\lambda^{(-)}(n) \\ 0 & 0 & \varepsilon_\lambda^{(-)}(1)2 & \varepsilon_\lambda^{(-)}(2)3 & \cdots & \varepsilon_\lambda^{(-)}(n-2)(n-1) & \varepsilon_\lambda^{(-)}(n-1)n \\ 0 & 0 & 0 & \varepsilon_\lambda^{(-)}(1)\binom{3}{2} & \cdots & \varepsilon_\lambda^{(-)}(n-3)\binom{n-1}{2} & \varepsilon_\lambda^{(-)}(n-2)\binom{n}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \varepsilon_\lambda^{(-)}(1)\binom{n-1}{n-2} & \varepsilon_\lambda^{(-)}(2)\binom{n}{n-2} \end{vmatrix}. \quad (2.25)$$

The recurrence relation (2.23) utilized for the above Hessenberg determinant $\mathbf{D}_n(\lambda|x)$ reproduces the defining property (2.5) of the polynomials $\tilde{\beta}_n(\lambda|x)$. In the classical $\lambda \rightarrow 0$ limit the above construction agrees to structure given in [7].

D. A two variable generalization

The two variable Kampè de Fèrìet generalization of the Hermite polynomials has been introduced [9, 10] via an extension of the generating function. Employing this method a new class of the Bernoulli polynomials has been obtained [11]. This technic has been recently used [12] to establish certain mixed special polynomial families associated with

Appell sequences. In the context of Tsallis type of deformation of the Bernoulli polynomials introduced here we now attempt the corresponding two variable extension via the following factorized generating function:

$$\frac{t}{\exp_\lambda(t) - 1} \exp_\lambda(tx) \exp_\lambda(t^r y) = \sum_{n=0}^{\infty} \tilde{\beta}_n^{(r)}(\lambda|x, y) \frac{t^n}{n!}, \quad \tilde{\beta}_n^{(r)}(\lambda|x, 0) = \tilde{\beta}_n(\lambda|x). \quad (2.26)$$

The recurrence relation for the $\tilde{\beta}_n^{(r)}(\lambda|x, y)$ polynomials follows by equating the coefficients of the identical powers of t^n on both sides of (2.26):

$$\begin{aligned} \tilde{\beta}_n^{(r)}(\lambda|x, y) &= \sum_{\ell=0}^{\lfloor \frac{n}{r} \rfloor} \frac{n!}{(n-r\ell)! \ell!} \varepsilon_\lambda^{(-)}(\ell) \varepsilon_\lambda^{(-)}(n-r\ell) x^{n-r\ell} y^\ell \\ &\quad - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{\ell} \varepsilon_\lambda^{(-)}(n+1-\ell) \tilde{\beta}_\ell^{(r)}(\lambda|x, y). \end{aligned} \quad (2.27)$$

We note that a null choice $y = 0$ in (2.27) readily reproduces the recurrence relation (2.5) observed for the single variable case. The recurrence relation (2.5) admits explicit solution of the polynomials $\tilde{\beta}_n^{(r)}(\lambda|x, y)$ as a double sum:

$$\begin{aligned} \tilde{\beta}_{n(\geq 0)}^{(r)}(\lambda|x, y) &= \sum_{j=0}^n \sum_{\ell=0}^{\lfloor \frac{n}{r} \rfloor} \frac{n!}{j! \ell! (n-j-r\ell)!} \varepsilon_\lambda^{(-)}(j) \varepsilon_\lambda^{(-)}(\ell) \beta_{n-j-r\ell}(\lambda) x^j y^\ell \\ &= \sum_{\ell=0}^{\lfloor \frac{n}{r} \rfloor} \frac{n!}{\ell! (n-r\ell)!} \varepsilon_\lambda^{(-)}(\ell) \tilde{\beta}_{n-r\ell}(\lambda|x) y^\ell, \end{aligned} \quad (2.28)$$

where the second equality provides the connection formula between the two variable and the single variable polynomials. The eigenfunction structure of the deformed exponential

$$\mathcal{D}_\lambda(y) \exp_\lambda(t^r y) = t^r \exp_\lambda(t^r y), \quad \mathcal{D}_\lambda(y) \equiv (1 + \lambda t^r y) \frac{\partial}{\partial y} \quad (2.29)$$

allows us to observe the differential recurrence properties of the polynomials $\tilde{\beta}_n^{(r)}(\lambda|x, y)$:

$$\frac{\partial \tilde{\beta}_n^{(r)}(\lambda|x, y)}{\partial x} = n \left(\tilde{\beta}_{n-1}^{(r)}(\lambda|x, y) - \lambda x \frac{\partial \tilde{\beta}_{n-1}^{(r)}(\lambda|x, y)}{\partial x} \right), \quad (2.30)$$

$$\frac{\partial \tilde{\beta}_n^{(r)}(\lambda|x, y)}{\partial y} = \frac{n!}{(n-r)!} \left(\tilde{\beta}_{n-r}^{(r)}(\lambda|x, y) - \lambda y \frac{\partial \tilde{\beta}_{n-r}^{(r)}(\lambda|x, y)}{\partial y} \right). \quad (2.31)$$

The differential structure (2.30, 2.31) provides the defining relations of the extension of λ -Appell sequences for the two variable polynomials.

III Conclusion

Altering the generating function of the Bernoulli polynomials via the Tsallis exponential function we have introduced a new deformation $\beta_n(\lambda|x)$ of the said polynomials. These

polynomials may be regarded as elements of the deformed λ -Appell sequence. Explicit expansion of the $\tilde{\beta}_n(\lambda|x)$ polynomials and the corresponding translation formula are derived. A representation of these polynomials via a Hessenberg type of determinantal structure has been provided. By augmenting the factorized generating function a two variable extension of these polynomials has been realized. Similar λ -deformations of, in particular, the Euler and Hermite polynomials may be of interest. A potential application of the polynomials discussed here lies in the area of nonextensive statistical mechanics [3] where the Tsallis exponential function plays a key role in the description of the entropy of the physical system. The polynomials presented here may be useful in developing a perturbative procedure for evaluation of such nonextensive statistical quantities. In particular, the investigations on the $(q-1)$ expansion [13] of various physical quantities valid asymptotically, may be facilitated by employing the $\tilde{\beta}_n(\lambda|x)$ polynomials.

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